Periodically Integrated Autoregression with a Structural Break

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Abstract

This paper considers a first order periodically integrated autoregressive (PIAR)model with a structural break. Firstly we introduce the PIAR model and the associated results obtained by Boswijk and Franses(1992). Then we extend the model to include a structural break and derive the asymptotic distribution the OLS estimator and the Likelihood ratio for testing periodic integration. Some remarks are made in connection with real quarterly data in Japan. The details of empirical studies will be reported at the conference of MODSIM95. In Appendix a simulation of Johansen's rank test is shown which is relevant to testing for periodic integration and cointegration in small sample.

1 Introduction

We consider a quarterly observed univariate time series $\{y_t, t = 1, \cdots, n\}$, simply denoted by $\{y_t\}$ in what follows. This process can be conveniently expressed in the multivariate representation by stacking the observation $\{y_t\}$ in the annual sequence of (4×1) vector

$$Y_T = (Y_{1T}, Y_{2T}, Y_{3T}, Y_{4T})',$$

where Y_{sT} is the observation in season s of year T,with $T = 1, ..., N$ and $N = n/4$. The multivariate series $\{Y_T\}$ is called as the vector of quarters (VQ) process of $\{y_t\}$. In economic time series it is often observed that each component in Y_T is random walk and there are cointegrating relationship among the four components. This phenomenon is modeled by the periodically integrated model developed by Boswijk and Franses(1993)(abbreviated as B&F hereafter). They gave a comprehensive treatment of such model. In this section we briefly reproduce their model and results as follows.

1.1 Model and Notations

B&F considered the periodic autoregressive process of order p for the above quarterly series $\{y_t\}$ with starting values $\{y_{1-p}, \cdots, y_0\}$. A periodic autoregressive process of order p (PAR(1)) is written as

$$y_t = \varphi_{1s} y_{t-1} + \cdots + \varphi_{ps} y_{t-p} + \varepsilon_t, \quad t = 1, ..., n, \ s = 1, ..., 4 \tag{1}$$

where φ_{is}, $i = 1, ..., p$ are periodically varying parameters, i.e. the coefficient of y_{t-1} equals φ_{is}, if time t corresponds to season s, and $\{\varepsilon_t\}$ is an independent $N(0, \sigma^2)$. Using the VQ representation the multivariate expression for (1) is given by

$$\Phi_0 Y_T = \Phi_1 Y_{T-1} + \cdots + \Phi_{T-P} + E_T,$$
$$T = 1, \ldots N, \tag{2}$$

where E_T is the VQ process of $\{\varepsilon_t\}$. Here Φ_i, $i = 0, \ldots, P$, are 4×4 parameter matrices. Let L denote the lag operator and define the matrix lag polynomial

$$\Phi(L) = \Phi_0 - \Phi_1 L - \cdots - \Phi_P L^P. \tag{3}$$

B&F defined the periodic integration as follows:

Definition (periodic integration)

The vector process $\{Y_T\}$ is stationary if the roots of the characteristic equation

$$\varphi(z) = |\Phi(z)| = 0 \tag{4}$$

are all outside the unit circle. In this case $\{y_t\}$ is said to be periodically stationary, denoted by $y_t \sim PI(0)$. If the characteristic equation has a single root equal to 1 and all other roots outside the unit circle, and $Y_{sT} \sim I(1)$ for all s, then $\{y_t\}$ is said to be periodically integrated of order 1, denoted by $y_t \sim PI(1)$.

It is known that the presence of a single unit root (and the assumption that all other roots are outside the unit circle) implies that $\{Y_T\}$ is cointegrated of order (1,1) with cointegrating rank 3 (See B&F).

In this paper we focus on the first-order periodic autoregression PAR(1):

$$y_t = \varphi_s y_{t-1} + \varepsilon_t \tag{5}$$

which can be expressed by the VQ representation

$$\Phi_0 Y_T = \Phi_1 Y_{T-1} + E_T$$

with

$$\Phi_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\varphi_2 & 1 & 0 & 0 \\ 0 & -\varphi_3 & 1 & 0 \\ 0 & 0 & -\varphi_4 & 1 \end{pmatrix},$$

$$\Phi_1 = \begin{pmatrix} 0 & 0 & 0 & \varphi_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic function is

$$\varphi(z) = |\Phi_0 - \Phi_1 z| = (1 - \varphi_1\varphi_2\varphi_3\varphi_4).$$

Hence $\{Y_T\}$ is stationary if

$$|\varphi_1\varphi_2\varphi_3\varphi_4| < 1,$$

and has a single unit root if

$$\Pi_{s=1}^4 \varphi_s = 1.$$

B&F proved that if $y_t \sim PI(1)$, then there exist constants φ_s, s=1,...,4, satisfying $\Pi_{s=1}^4 \varphi_s = 1$, such that the difference $u_t = y_t - \varphi_s y_{t-1}$ is periodically stationary. They called the series as a periodically integrated autoregression of order p, denoted $PIAR(1,1)$. Because of this theorem we can rewrite, by backward substitution,

$$y_t = \varphi_s y_{t-1} + u_t$$

$$= \varphi_s\varphi_{s-1} y_{t-2} + u_t + \varphi_s u_{t-1}$$

$$\cdots\cdots$$

$$= y_{t-4} + u_t + \varphi_s u_{t-1} + \varphi_s\varphi_{s-1} u_{t-2}$$

$$+ \varphi_s\varphi_{s-1}\varphi_{s-2} u_{t-3},$$

so that $\Delta_4 y_t$ is a periodic moving average of order 3 in u_t (where $\Delta_q = (1 - L^q)$, the qth difference operator).The VQ process of $\Delta_4 y_t$ is $\Delta_1 Y_T$, which has the following vector moving average (VMA) representation:

$$\Delta_1 Y_T = (\Theta_0 + \Theta_1 L)E_T,$$

where

$$\Theta_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \varphi_2 & 1 & 0 & 0 \\ \varphi_2\varphi_3 & \varphi_3 & 1 & 0 \\ \varphi_2\varphi_3\varphi_4 & \varphi_3\varphi_4 & \varphi_4 & 1 \end{pmatrix},$$

$$\Theta_1 = \begin{pmatrix} 0 & \varphi_3\varphi_4\varphi_1 & \varphi_4\varphi_1 & \varphi_1 \\ 0 & 0 & \varphi_4\varphi_1\varphi_2 & \varphi_1\varphi_2 \\ 0 & 0 & 0 & \varphi_1\varphi_2\varphi_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that we can write $\Theta_0 + \Theta_1 = ab'$ where

$$a = \begin{pmatrix} 1 \\ \varphi_2 \\ \varphi_2\varphi_3 \\ \varphi_2\varphi_3\varphi_4 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ \varphi_3\varphi_4\varphi_1 \\ \varphi_4\varphi_1 \\ \varphi_1 \end{pmatrix} \tag{6}$$

1.2 TESTING FOR $\Pi_{s=1}^4 \varphi_s = 1$.

Now we extend the PAR(1) to include constants and trends. As the most general expression, B&F considered

$$y_t = \varphi_s y_{t-1} + \alpha_s + \beta_s T_s + \varepsilon_t. \tag{7}$$

where T_t is a step function that signifies the year in period t. The corresponding VQ representation is

$$\Phi_0 Y_T = \Phi_1 Y_{T-1} + \alpha + \beta T + E_T$$

where

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)'$$
and
$$\beta = (\beta_1, \beta_2, \beta_3, \beta_4)'.$$

The VMA representation for this model is

$$\Delta_1 Y_T = (\Theta_0 + \Theta_1 L)(\alpha + \beta T + E_T)$$

$$= (\Theta_0 + \Theta_1)\alpha - \Theta_1\beta + (\Theta_0 + \Theta_1)\beta T$$

$$+(\Theta_0 + \Theta_1 L)E_T.$$

Substituting a,b we can rewrite the VMA representation as

$$\Delta_1 Y_T = ab'\alpha - \Theta_1\beta + ab'\beta T$$

$$+(\Theta_0 + \Theta_1 L)E_T.$$

We can rewrite this mode as

$$y_t = \textstyle\sum_{s-1}^4 \varphi_s D_{st} y_{t-1} + \sum_{s-1}^4 \alpha_s D_{st}$$

$$+ \textstyle\sum_{s-1}^4 \beta_s D_{st} T_t + \varepsilon_t, \tag{8}$$

where D_{st} is a seasonal dummy variable such that $D_{st} = 1$ if time t is in season s and D_{st}=0 elsewhere. B&F considered the likelihood ratio test statistic

$$LR_i = n(\ln \tilde{\sigma}_i^2 - \ln \hat{\sigma}_i^2)$$

for testing

$$\pi = \Pi_{s=1}^4 \varphi_s = 1$$

in the case (i) as in Theorem 1 below, where $\hat{\sigma}_i^2$ is the variance estimator obtained by OLS residual in (8) and $\tilde{\sigma}_i^2$ is the variance estimator by obtained ML residual under the restriction $\pi = 1$ in (8). They show the following result.

Theorem 1. (Boswijk and Franses 1992): *Let $\{y_t\}$ be generated by the periodic autoregression (8), and let b as given in (6). Then we have, under H_0 : $\Pi_{s=1}^{4}\varphi_s = 1$, and $n \to \infty$,*

$$LR_i \Rightarrow$$

$$\left(\left(\int_0^1 W_i(r)^2 dr\right)^{-\frac{1}{2}} \int_0^1 W_i(r) dW(r)\right)^2, \quad (9)$$

where $W(r)$ is a standard Brownian motion process, $\Rightarrow$ denotes weak convergence in distribution and

(i) if $\alpha = 0$ and $\beta = 0$, then

$$W_1(r) = W(r);$$

(ii) if $b'\alpha = 0$ and $\beta = 0$, then

$$W_2(r) = W(r) - \int_0^1 W(t)dt;$$

(iii) if $b'\beta$, then

$$W_3(r) =$$

$$W_2(r) - 12(r - \tfrac{1}{2}) \int_0^1 (t - \tfrac{1}{2}) W_2(t) dt.$$

They also prove the asymptotic result for the case (i):

$$N(\hat{\pi} - 1) \Rightarrow$$

$$\left[\int_0^1 W(r)^2 dr\right]^{-1} \int_0^1 W(r) dW(r).$$

which is the same asymptotic null distribution as Fuller's(1976) $n(\hat{\rho} - 1)$ for a unit root in a (non-periodic) autoregression.

2 Structural Break

In recent econometric literatures the unit root problem is discussed in association with structural change in non-periodic time series after Perron(1989) was published. In this section we deal with this problem in periodic time series. First we extend the PIAR(1,1) model to include a structural break.

Let $I_{(t \geq \tau)}$ and g_τ denote an indicator function such that

$$I_{(t \geq \tau)} = 0 \text{ if } t < \tau,$$

$$= 1 \text{ if } t \geq \tau,$$

and

$$g_\tau = 0 \qquad \text{if } t < \tau,$$

$$= t - \tau \quad \text{if } t \geq \tau.$$

Consider the following two cases:

$$(iv) \quad y_t = \textstyle\sum_{s=1}^{4} \varphi_s D_{st} y_{t-1}$$

$$+ \textstyle\sum_{s=1}^{4} \alpha_s D_{st} + \sum_{s=1}^{4} \beta_s D_{st} I_{(t \geq \tau)} + \varepsilon_t, \quad (10)$$

$$(v) \quad y_t = \textstyle\sum_{s=1}^{4} \varphi_s D_{st} y_{t-1}$$

$$+ \textstyle\sum_{s=1}^{4} \alpha_s D_{st} + \sum_{s=1}^{4} \beta_s D_{st} g_\tau + \varepsilon_t, \quad (11)$$

Note that a break point in time τ is assumed to be known. Furthermore as in Perron(1989), we assume that a ratio

$$\lambda = \frac{N_1}{N}$$

is fixed as $n \to \infty$ where N is the number of years and N_1 is the number of years before a structural break. This assumption is not realistic but indispensable to develop the asymptotic theory. For the two cases we have

Theorem 2. Let LR_4 denote the likelihood ratio in the case (iv) when $b'\beta = 0$, then we have, under $H_0 : \Pi_{s=1}^4 \varphi_s = 1$, $n \to \infty$,

$$N(\hat{\pi} - 1) \Rightarrow$$

$$\left[\int_0^1 \tilde{W}(r)^2 dr\right]^{-1} \int_0^1 \tilde{W}(r) dW(r),$$

$$LR_4 \Rightarrow$$

$$\left\{\left[\int_0^1 \tilde{W}(r)^2 dr\right]^{-\frac{1}{2}} \int_0^1 \tilde{W}(r) dW(r)\right\}^2,$$

where

$$\tilde{W}(r) = (1 - I_{(t \geq \tau)})\tilde{W}_1(r) + I_{(t \geq \tau)}\tilde{W}_2(r).$$

with

$$\tilde{W}_1(r) = W(r) - \int_0^\lambda W(r) dr$$

$$\tilde{W}_2(r) = W(r) - \int_\lambda^1 W(r) dr$$

$$\lambda = \frac{N_1}{N} \quad and \quad \lim_{n \to \infty} \lambda = k,$$

where k is a constant and $W(r)$ is a standard Wiener Process.

Proof. given in Appendix B which is available from the author upon request.

Theorem 3. Let LR_5 denote the likelihood ratio in the case (v) when $b'\beta = 0$, then we have, under $H_0 : \Pi_{s=1}^4 \varphi_s = 1$, $n \to \infty$,

$$N(\hat{\pi} - 1) \Rightarrow$$

$$\left[\int_0^1 \hat{W}(r)^2 dr\right]^{-1} \int_0^1 \hat{W}(r) dW(r),$$

$$LR_5 \Rightarrow$$

$$\left\{\left[\int_0^1 \hat{W}(r)^2 dr\right]^{-\frac{1}{2}} \int_0^1 \hat{W}(r) dW(r)\right\}^2,$$

where

$$\hat{W}(r) = W_3(r)$$

$$-h(\lambda)^{-1} n_\lambda(r) \int_0^1 n_\lambda(r) W(r) dr$$

with

$$h(\lambda) = \tfrac{1}{3}\lambda^3(1-\lambda)^3,$$

$$n_\lambda(r) = \max(r - 1 + \lambda, 0) - \tfrac{\lambda^2}{2}$$

$$-(3\lambda^2 - 2\lambda^3)(r - \tfrac{1}{2}),$$

where

$$\lambda = \tfrac{N_2}{N} \quad with \quad N_2 = N - N_1 \ and$$

$$\lim_{n \to \infty} \lambda = k,$$

where k is a constant.

Proof. given in Appendix B which is available from the author upon request.

The expression $h(\lambda)$ and $n_\lambda(r)$ are the same as in Hatanaka and Koto(1993) who dealt with structural change in non-periodic time series. They also deal with the same problem in their 1995 paper.

The non-standard asymptotic results in *Theorems 1,2,3* implies that the standard test procedures are not valid. To make these asymptotic distributions useful for a practical purpose it is necessary to tabluate the distribution by simulation. Franses and McAleer(1994) considered an alternative testing procedures based on a generalized difference $y_t - \hat{\varphi}_s y_{t-1}$,and hence their test can be treated within the standard asymptotic theory.

3 Examples and Remarks

In order to see if there are periodic integration in real quarterly data we have to perform the following test procedures :(1) Unit root test for each series of Y_{sT}, in sth quarter, s=1,...,4, by, say, Dickey-Fuller test, (2) Periodic unit root test or testing for the hypothesis H_0: $y_t \sim PI(1)$ based on the previously obtained result that if $y_t \sim PI(1)$, then a statistic $N(\hat{\pi}-1)$ the same asymptotic distribution as Dickey-Fuller test statistic $n(\hat{\rho}-1)$. Or (2') Cointegration rank test for the hypothesis H_0:$\{Y_T\}$ is cointegrated of order (1,1) with cointegrating rank 3. If $y_t \sim PI(1)$, then a statistic $N(\hat{\pi}-1)$ the same asymptotic distribution as Dickey-Fuller test statistic $n(\hat{\rho}-1)$ and $\{Y_T\}$ is cointegrated of order (1,1) with cointegrating rank 3.

Statistical tables for these tests are given in literatures such as Fuller(1976), Johansen(1988), and Osterwald-Lunum(1992). But we have to be careful to use tables for Johansen's rank test because they are based on asymptotic distribution. As the sample size N, a number of year, of most quarterly time series data are not large enough to apply these tables. We simulated 95% critical point of Johansen's rank test for sample size N=1000,500,100,50, and 25 and the results are shown in Appendix. It shows that the asymptotic critical points of Johansen's rank test are inappropriate when sample size is very small such as $N = 25$ which is often the case in empirical studies.

As an example we examine the quarterly time series of Japanese GNP for 1970.I-1992.IV. In this case we verified that each quarter was random walk and obtained $N(\hat{\pi}-1) = 0.87$ indicating that the quarterly GNP series $\{y_t\}$ is periodically integrated. Although the VQ process $\{Y_T\}$ should have rank 3 in theory if $\{y_t\} \sim PI(1)$, Johansen's rank test could not show that rank=3.

So far we have dealt with a quarterly time series ($s = 4$) , but the idea of periodic integration can be extended to any s. As an another example we consider the daily FTSE index of UK for from 31 December, 1979 to 17 April, 1995 in which $s = 5$, or five days of the week are regraded as 5 seasons. In this case VQ process is replaced with a vector of days (VD)

$$Y_T = (Y_{1T}, Y_{2T}, Y_{3T}, Y_{4T}, Y_{5T})'.$$

We verified that each day of the week was random walk. A statistic $N(\hat{\pi}-1)$ can be used to test the null of periodic integration $PI(1)$ only when $s = 4$. Therefore we used Johansen's rank test for testing the rank of Y_T and found that the rank was 4, implying that the daily FTSE index is periodically integrated of $PI(1)$.

The detailed results of this example and other empirical study will be shown at the conference of MODSIM95.

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Appendix A:
Simulation of Johansen's Rank Test of Cointegration in Small Sample.

Johansen's Maximum Likelihood Cointegration Rank Test Statistic are simulated for sample size 1000, 500, 100, 50, and 25. We generated artificial VQ series $\{Y_T\}$ for the four cases with cointegration rank r=0,1,2, and 3. The restricted and unrestricted likelihood ratios were calculated from the two auxiliary kth order VAR regressions:

$$\Delta y_t = \hat{\pi}_0 + \hat{\Pi}_1 \Delta y_{t-1} + \hat{\Pi}_2 \Delta y_{t-2} + \cdots + \hat{\Pi}_k \Delta y_{t-k-1} + \hat{u}_t,$$
$$y_{t-1} = \hat{\theta}_0 + \hat{\Theta}_1 \Delta y_{t-1} + \hat{\Theta}_2 \Delta y_{t-2} + \cdots + \hat{\Theta}_k \Delta y_{t-k-1} + \hat{v}_t$$

where $\hat{\pi}_0$, $\hat{\Pi}_j$, $\hat{\theta}_0$, $\hat{\Theta}_j$ are the OLS estimators and $\hat{u}_t$ and $\hat{v}_t$ are OLS residuals. In our simulation k was chosen 1,2,3,5,and 10. The procedures of calculations are summarized in Ch.20 of Hamilton(1994). Only a part of the simulation results are given in the next table.

Simulated 95% Critical Points of Johansen Rank Test

	Cointegration Rank	T= 1000	500	100	50	25
k=1	0	40.26	40.44	43.11	46.83	56.98
	1	24.45	24.57	25.73	25.58	27.34
	2	12.59	12.53	12.63	12.36	11.68
	3	2.38	2.35	2.39	2.44	2.65
k=2	0	40.4	40.79	44.4	50.7	81.61
	1	24.23	24.35	25.74	26.83	34.71
	2	12.16	12.43	12.77	12.19	13.13
	3	2.34	2.45	2.55	2.57	3.18
k=3	0	40.84	41.26	46.72	58.16	167.77
	1	24.55	24.87	26.35	28.1	64.32
	2	12.42	12.45	12.76	12.07	20.86
	3	2.46	2.39	4.49	2.69	4.79
k=5	0	41.35	41.81	52.05	81.2	NA
	1	24.68	25.07	27.28	38.35	NA
	2	12.12	12.68	12.59	15.25	NA
	3	2.34	2.35	2.72	4.19	NA
k=10	0	42.11	43.82	75.59	197.77	NA
	1	24.6	25.77	33.34	138.19	NA
	2	12.56	12.58	12.75	68.89	NA
	3	2.5	2.51	3.06	21.52	NA

Corresponding Asymptotic Values for T=∞

0	38.89
1	24.31
2	12.53
3	3.84

T=sample size(number of years of quarterly data).
k=number of lags in the auxiliary regressions.
Asmptotic values are cited from Osterwald-Lenum.